General Form of System Dynamics Models

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The General Notation of Hayward & Roach (2017)

In appendix B of Hayward & Roach (2017) there is the following description of a system dynamics model:

> Consider an nth order system dynamics model with stocks x_i, $i = 1, \ldots, n$ with net flows f_i, and with π_{ij} causal pathways from x_i to x_j. Let a_{ij} be the names of the list of causal pathways from x_i to x_j. Thus a_{ij} is a matrix of lists of possibly differing lengths π_{ij}. An individual causal pathway in the list is indexed by μ_{ij} drawn from the range 1 to π_{ij}, thus giving the matrix of lists $a_{ij\,\mu_{ij}}$, which can be abbreviated to $a_{ij\,\mu}$ without confusion. Each element of each list is a collection of intermediary auxiliary variables in pathway μ_{ij}. The nth order system dynamics model in a concise networked equation form is:
>
> $$\dot{x}_i \;=\; f_i(x_{j\underline{a_{ji\,\mu}}}) \quad i, j = 1, \ldots, n; \;\; \mu_{ji} = 1, \ldots, \pi_{ji} \qquad (1)$$
>
> $x_{j\underline{a_{ji\,\mu}}}$ is the variable x_j along pathway $a_{ji\,\mu} \equiv a_{ji\,\mu_{ji}}$ connected to x_i. There are π_{ji} pathways connecting these variables[1].

It is possible some people are unfamiliar with the very terse notation used by mathematicians when they generalise mathematical concepts. In the case of the appendix in Hayward & Roach (2017), this confusion may be compounded with the use of the underlined index to label pathways between adjacent stocks.

A few notes first:

- The $f_i(x_{j\underline{a_{ji\,\mu}}})$ with the pathway label removed is just $f_i(x_j)$, which is a standard shorthand for $f_i(x_1, x_2, \ldots, x_n)$. That is, the flow f_i for stock x_i is potentially a function of all the stocks in the system. In a system dynamics model functional dependence is indicated using connectors and maybe a number of intermediate converters.
- The underline index $a_{ji\,\mu}$ allows for multiple pathways between two stocks, labelled by i and j, though paths will have different intermediary converters. (This notation

[1] A typo in the paper has been corrected. The paper had π_{jj} at the end of equation (1). It should be π_{ji}, as given here, to match the μ_{ji}.

allows for multiple pathways from a stock to itself, $a_{ii\,\underline{\mu}}$.) $a_{ji\,\underline{\mu}}$ is a label name for the pathway from stock x_j – the source stock, to x_i – the target stock. The pathway label, $a_{ji\,\underline{\mu}}$, could be a loop name, or the names of all the intermediary converters between source and target stock, which will include a flow on the target x_i. The label could be given any other suitable name. Loop names may not be appropriate if they share connections between stocks.

- The pathway index is underlined to distinguish it from indices that indicate different variables, which have no underline.

- π_{ji} is the number of connections between a pair of stocks. μ_{ji} indexes the a_{ji} allowing each connection to have a different label. The j and the i are in reverse order as the equation labelled by i is for the target stock, the left hand side of the equation. Thus i must represent the target stock, for various sources j. The first index of a is the source, and its second index is the target, because it is natural to read from left to right in order of causality. The source stocks, the causes, are on the right-hand side of the equation.

Of course all this is still all rather abstract. What I will give next is an example of a system dynamics model and relate its elements to the general form quoted above.

Example System Dynamics Model

I have chosen a two stock system with a number of connections, and as such, a number of loops. Thus $n = 2$. It is a hypothetical model - not representative of anything in the real world.

This is the system expressed by its underlying differential equations for two stocks (dynamical variables) $x(t)$ and $y(t)$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = bxy - cx \qquad (2)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = dxy - e\frac{xy}{1+x} - mx \qquad (3)$$

Equations (2–3) are very much the form a system may appear in a book or paper on mathematical biology or population modelling. It would be a familiar form to anyone who has taken a course in non-linear dynamical systems.

To move towards the form in (1) the x and y are replaced by x_1 and x_2 respectively. Thus, the subscripts 1 and 2 now distinguish the stocks. These subscript values are contained in the index i in (1), hence two equations (2–3); and in the index j, implying the right-hand sides of the two equations are functions of two variables. Thus, $i, j = 1, 2$ means i and j take the two values in this range. Given that interpretation then (1) stands for two equations, and the flows f_1 and f_2 depend on both stocks x_1 and x_2.

Equations (2–3) are now rewritten as:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = bx_1x_2 - cx_1 \tag{4}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = dx_1x_2 - e\frac{x_1x_2}{1+x_1} - mx_1 \tag{5}$$

Of course, a system dynamics modeller does not start with differential equations but with a stock/flow diagram with equations contained in the different model elements. Thus I will say that my starting point is the model that is given in figure 1.

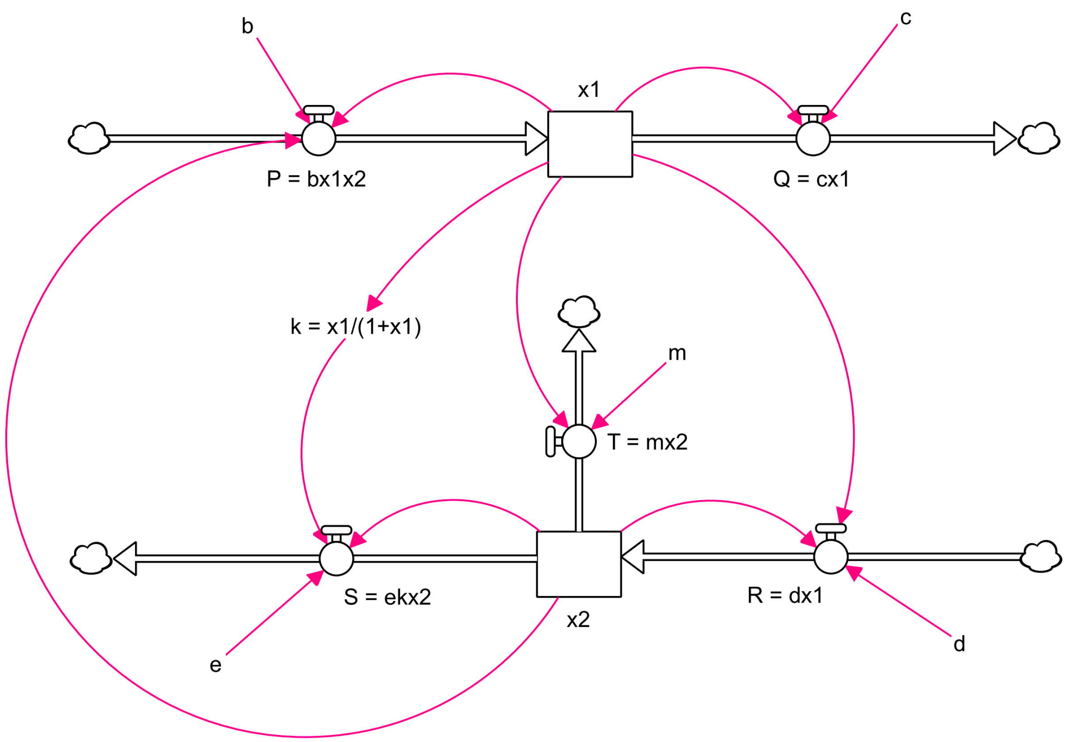

Fig. 1: Example System Dynamics Model to Illustrate General Notation.

The notation in figure 1 is a bit unusual for a system dynamics modeller. The stocks, flows and converters would usually have names rather than symbols to indicate the concepts they represent. But here the symbols are used to ensure the equations and formulae follow mathematical conventions. Also, I have included the formulae of the converters and flows along with their name. Thus, inflow P on stock x_1 is determined by $P = bx_1x_2$. Normally a system dynamics model is presented with the equations "hidden" within the software, either extracted by a mouse click, or by giving a complete equation listing. Thus, a modeller normally never sees all aspects of the systems dynamics model at once. But that does not help the abstraction process. Thus, figure 1 is both the stock/flow model and the model equations[2].

[2]The only formulae missing from figure 1 are the parameter values and initial conditions. However, these are not needed until a model is solved numerically. Figure 1 contains all the model structure.

The system dynamics model figure 1 can be reduced to the differential equations (4–5) by using substitution to eliminate the intermediate variables between the stocks:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1, x_2) = P - Q = bx_1x_2 - cx_1 \quad (6)$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1, x_2) = R - S - T = R - ekx_2 - T = dx_1x_2 - e\frac{x_1x_2}{1+x_1} - mx_1 \quad (7)$$

The problem with this form of the equations, (6–7), is that the information about all the causal connections between stocks is lost. These causal connections are part of feedback loops, indicating the impacts the stocks have on each other due to the different processes in the model assumptions. These causal connections are essential for loop impact analysis and its interpretation in terms of forces. For example, looking at (7) the source stock x_1 occurs four times. However, it is not clear from the equation that there are three casual pathways from x_1 to target stock x_2 without reference to figure 1. The process of reducing the system dynamics model to differential equations has lost information about the model.

Instead, I will annotate the source stock names on the right-hand sides of (6–7) with the pathway name that leads it to the target stocks on their left-hand side. In this case I will use the names of the intermediary converters. Thus, the pathway label from source stock x_1 to the inflow R of target stock x_2 is $\underline{R}$. Thus, one of the x_1s on the right-hand side of 7 will be written $x_{1\underline{R}}$. However, the pathway label from x_1 to outflow S of x_2 is $\underline{kS}$, because the converter k is in the path. Thus, another of the x_1s will be written $x_{1\underline{kS}}$.

Using the above labelling, the equations that fully represent the model in figure 1 are:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = bx_{1\underline{P}}x_{2\underline{P}} - cx_{1\underline{Q}} \quad (8)$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = dx_{1\underline{R}}x_{2\underline{R}} - e\frac{x_{1\underline{kS}}x_{2\underline{S}}}{1+x_{1\underline{kS}}} - mx_{1\underline{T}} \quad (9)$$

Equations (8–9) are the specific equivalent of the general form (1) that appeared in appendix B of Hayward & Roach (2017). We referred to them as causally connected differential equations.

1. In (8) there are two versions of x_1, i.e. $x_{1\underline{P}}$ and $x_{1\underline{Q}}$, because there are two connections from x_1 to itself in figure 1.

2. In (8) there is only one version of x_2, i.e. $x_{2\underline{P}}$, because there is only one connection from x_2 to x_1 in figure 1.

3. In (9) there are three versions of x_1, i.e. $x_{1\underline{R}}$, $x_{1\underline{kS}}$ and $x_{1\underline{T}}$, because there are three connections from x_1 to x_2 in figure 1.

4. In (9) there are two versions of x_2, i.e. $x_{2\underline{R}}$ and $x_{2\underline{S}}$, because there are two connections from x_2 to itself in figure 1.

Connecting the Example to the General Form

I have already pointed out that equations (8–9) are the specific equivalent of the general form (1) . From the bullets points 1–4 the symbol π_{ij} can be evaluated, the number of connections from stock x_i to stock x_j. Thus:

$$\begin{aligned}
\pi_{11} &= 2 \quad \text{bullet point 1} && (10)\\
\pi_{12} &= 3 \quad \text{bullet point 3} && (11)\\
\pi_{21} &= 1 \quad \text{bullet point 2} && (12)\\
\pi_{22} &= 2 \quad \text{bullet point 4} && (13)
\end{aligned}$$

Thus π is a 2×2 matrix:

$$\pi = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \tag{14}$$

That bullets 3 and 2 are not in order in (11–12) also helps explain why the indices of π in (1) are in reverse order. Matrices are written in row followed by column order, which represents a source followed by its target. However, a single equation, for example (9), represents one target indexed by i in (1), which is the column index of π_{ji}. The right-hand side of the equation lists all the target stock's sources, the row index of π_{ji}.The matrices have causality running from left to right, but the equations have causality operating from right-hand side to left!

The pathway labels $a_{ij\mu}$ are lists of pathway names:

$$\begin{aligned}
a_{11\,\mu} \equiv a_{11\,\mu_{11}} &= (P, Q) \quad \text{from } x_1 \text{ to itself} && (15)\\
a_{12\,\mu} \equiv a_{12\,\mu_{12}} &= (R, kS, T) \quad \text{from } x_1 \text{ to } x_2 && (16)\\
a_{21\,\mu} \equiv a_{21\,\mu_{21}} &= (P) \quad \text{from } x_2 \text{ to } x_1 && (17)\\
a_{22\,\mu} \equiv a_{22\,\mu_{22}} &= (R, S) \quad \text{from } x_2 \text{ to itself} && (18)
\end{aligned}$$

The μ_{ij} are just the counters in the lists in (15–18). For example, $\mu_{12} = 1, \ldots, 3$ runs through the 3 slots in the list (R, kS, T). That is, μ_{12} runs through the 3 pathways from x_1 to x_2, thus through the three forces of x_1 on x_2. This notation is helpful in extracting the impacts of stocks on each other from the equations. The range can be different in each matrix element, because a_{ij} is a matrix of lists. The lists can be thought of as vectors when applied to impacts.

Thus, a can be put in matrix form:

$$a = \begin{bmatrix} (P, Q) & (R, kS, T) \\ (P) & (R, S) \end{bmatrix} \tag{19}$$

(19) is just a matrix of pathway labels. It looks a bit unorthodox as it is a matrix whose elements are vectors of differing dimension. This construct is well defined, but they were called lists rather than vectors in Hayward & Roach (2017) to avoid dealing with its formal definition, which was beyond the scope of that paper.

The row number of (19) is the the index j of equation (1). i is the a's column number, and μ is the position number in the vector at a given matrix position, where μ has a different range from 1 to the number in the array for π, (14). Both μ and π are subscripted with j and i to select the correct array position. Thus, for example, for $j = 1$ and $i = 2$ in $a_{ji\,\mu}$, then $a_{12\,1} = R$, $a_{12\,2} = kS$ and $a_{12\,3} = T$, the three elements of the top right vector in the matrix of (19). These elements represent the three separate causal pathways from x_1 to x_2.

What is a Pathway Subscripted Variable?

The pathway notation is able to capture the effects of one stock on another and identify the change imparted along each causal pathway. However, it does mean that a variable, such as x_1 in the differential equations (4–5) appears in various pathway forms, such as $x_{1\underline{P}}$ and $x_{1\underline{kS}}$, on the right hand side of the causally connected differential equations (8–9). What is the difference between the two notations, for example, between x_1 and $x_{1\underline{P}}$?

The x_1 refers only to the numerical value of the variable, i.e. its value as a function of time t. The version with the pathway subscript, e.g. $x_{1\underline{P}}$, has information on both the value of the variable and its particular causal origin, via P. Thus, $x_{1\underline{P}}$ and $x_{1\underline{Q}}$ have the same value, i.e. x_1, which depends on time, but different causal origins, P and Q, which do not depend on time. It would be straightforward to construct functions that would extract the value and the causal origin from a pathway subscripted variable.

Reference

Hayward J. & Roach P.A. (2017). Newton's Laws as an Interpretive Framework in System Dynamics. System Dynamics Review, 33(3-4), 183-218. DOI: 10.1002/sdr.1586.